# ON THE EXISTENCE OF CONTINUOUS SELECTIONS AVOIDING EXTREME POINTS (Nonlinear Analysis and Convex Analysis)



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# ON THE EXISTENCE OF CONTINUOUS SELECTIONS AVOIDING EXTREME POINTS

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Throughout this note, all spaces are assumed to be $T_1$ and $\lambda$ stands for an infinite cardinal number. For undefined terminology, we refer to [3]. The purpose of this note is to introduce some results of [15] and [16].

Let $X$ be a space and $(Y, \|\cdot\|)$ a Banach space. By $2^Y$, we denote the set of all non-empty subsets of $Y$. For a mapping $\varphi : X \to 2^Y$, a mapping $f : X \to Y$ is called a *selection* if $f(x) \in \varphi(x)$ for each $x \in X$.

For $K \in \mathcal{F}_c(Y)$, a point $y \in K$ is called an *extreme point* if every open line segment containing $y$ is not contained in $K$. For $K \in \mathcal{F}_c(Y)$, the *weak convex interior* $\mathrm{wci}(K)$ of $K$ ([5]) is the set of all non-extreme points of $K$, that is,

$$\mathrm{wci}(K) = \{y \in K \mid y = \delta y_1 + (1-\delta) y_2 \text{ for some } y_1, y_2 \in K \setminus \{y\} \text{ and } 0 < \delta < 1\}.$$

Our concern of this note is to obtain theorems on continuous selections avoiding extreme points, which is motivated by Problem 3 below posed by V. Gutev, H. Ohta and K. Yamazaki [5].

## 1. A problem of Gutev, Ohta and Yamazaki

A Hausdorff space $X$ is called *countably paracompact* if every countable open cover of $X$ is refined by a locally finite open cover of $X$. Let $\mathbf{R}$ be the space of real numbers with the usual topology. The following insertion theorem due to C. H. Dowker [2] and M. Katětov [7] is fundamental in our study.

**Theorem 1** (Dowker [2, Theorem 4], Katětov [7, Theorem 2]). *A $T_1$-space $X$ is normal and countably paracompact if and only if for every upper semicontinuous function $g : X \to \mathbf{R}$ and every lower semicontinuous function $h : X \to \mathbf{R}$ with $g(x) < h(x)$ for each $x \in X$, there exists a continuous function $f : X \to \mathbf{R}$ such that $g(x) < f(x) < h(x)$ for each $x \in X$.*

The cardinality of a set $S$ is denoted by $\mathrm{Card}\, S$. A $T_1$-space $X$ is called $\lambda$-*collectionwise normal* if for every discrete collection $\{F_\alpha \mid \alpha \in A\}$ of closed subsets of $X$ with $\mathrm{Card}\, A \leq \lambda$, there exists a disjoint collection $\{G_\alpha \mid \alpha \in A\}$ of open subsets of $X$ such that $F_\alpha \subset G_\alpha$ for each $\alpha \in A$. The space $c_0(\lambda)$ is the Banach space consisting of functions $s : D(\lambda) \to \mathbf{R}$, where $D(\lambda)$ is a set with $\mathrm{Card}\, D(\lambda) = \lambda$, such that for each $\varepsilon > 0$ the set $\{\alpha \in D(\lambda) \mid |s(\alpha)| \geq \varepsilon\}$ is finite, where the linear operations are defined pointwise and $\|s\| = \sup\{|s(\alpha)| \mid \alpha \in D(\lambda)\}$ for each $s \in c_0(\lambda)$. In order to connect insertion theorems with selection theorems, V. Gutev, H. Ohta and K. Yamazaki [5] introduced lower and upper semicontinuity of a mapping to the Banach space $c_0(\lambda)$ and, with the aid of these concepts, they proved sandwich-like characterizations of paracompact-like properties. Moreover, they introduced generalized $c_0(\lambda)$-spaces for Banach spaces and established the following theorem. A mapping $\varphi : X \to 2^Y$ is called *lower semicontinuous* (*l.s.c.* for short) if for every open subset $V$ of $Y$, the set $\varphi^{-1}[V] = \{x \in X \mid \varphi(x) \cap V \neq \emptyset\}$ is open in $X$. By $\mathcal{C}_c(Y)$ we denote the set of all non-empty compact convex subsets of $Y$ and let $\mathcal{C}'_c(Y) = \mathcal{C}_c(Y) \cup \{Y\}$.

**Theorem 2** (Gutev, Ohta and Yamazaki [5, Theorem 4.5]). *A $T_1$-space $X$ is countably paracompact and $\lambda$-collectionwise normal if and only if for every generalized $c_0(\lambda)$-space $Y$, every l.s.c. mapping $\varphi : X \to \mathcal{C}'_c(Y)$ with* $\operatorname{Card} \varphi(x) > 1$ *for each $x \in X$ admits a continuous selection $f : X \to Y$ such that $f(x) \in \operatorname{wci}(\varphi(x))$ for each $x \in X$.*

Note that the "only if" part of Theorem 2 implies that of Theorem 1. By $w(Y)$ we denote the weight of a space $Y$. Since generalized $c_0(\lambda)$-space is a special Banach space with $w(Y) \leq \lambda$, Gutev, Ohta and Yamazaki [5] posed the following problem.

**Problem 3** (Gutev, Ohta and Yamazaki [5, Problem 4.7]). *Can the phrase "every generalized $c_0(\lambda)$-space $Y$" in Theorem 2 be replaced with "every Banach space $Y$ with $w(Y) \leq \lambda$"?*

It is proved in [15] that the answer of Problem 3 is affirmative.

**Theorem 4** ([15]). *A $T_1$-space $X$ is countably paracompact and $\lambda$-collectionwise normal if and only if for every Banach space $Y$ with $w(Y) \leq \lambda$, every l.s.c. mapping $\varphi : X \to \mathcal{C}'_c(Y)$ with* $\operatorname{Card} \varphi(x) > 1$ *for each $x \in X$ admits a continuous selection $f : X \to Y$ such that $f(x) \in \operatorname{wci}(\varphi(x))$ for each $x \in X$.*

In particular, we have the following.

**Corollary 5.** *A $T_1$-space $X$ is countably paracompact and normal if and only if for every separable Banach space $Y$, every l.s.c. mapping $\varphi : X \to \mathcal{C}'_c(Y)$ with* $\operatorname{Card} \varphi(x) > 1$ *for each $x \in X$ admits a continuous selection $f : X \to Y$ such that $f(x) \in \operatorname{wci}(\varphi(x))$ for each $x \in X$.*

**Corollary 6.** *A $T_1$-space $X$ is countably paracompact and collectionwise normal if and only if for every Banach space $Y$, every l.s.c. mapping $\varphi : X \to \mathcal{C}'_c(Y)$ with* $\operatorname{Card} \varphi(x) > 1$ *for each $x \in X$ admits a continuous selection $f : X \to Y$ such that $f(x) \in \operatorname{wci}(\varphi(x))$ for each $x \in X$.*

A Hausdorff space $X$ is called *$\lambda$-paracompact* if every open cover $\mathcal{U}$ of $X$ with $\operatorname{Card} \mathcal{U} \leq \lambda$ is refined by a locally finite open cover of $X$. The set of all non-empty closed convex subsets of a Banach space $Y$ is denoted by $\mathcal{F}_c(Y)$. The following theorem is a $\lambda$-paracompact analogue of Theorems 2 and 4.

**Theorem 7** ([15]). *A $T_1$-space $X$ is normal and $\lambda$-paracompact if and only if for every Banach space $Y$ with $w(Y) \leq \lambda$, every l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$ with* $\operatorname{Card}\varphi(x) > 1$ *for each $x \in X$ admits a continuous selection $f : X \to Y$ such that $f(x) \in \operatorname{wci}(\varphi(x))$ for each $x \in X$.*

Thus we have the following variation of [11, Theorem 3.2″].

**Corollary 8.** *A $T_1$-space $X$ is paracompact if and only if for every Banach space $Y$, every l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$ such that* $\operatorname{Card}\varphi(x) > 1$ *for each $x \in X$ admits a continuous selection $f : X \to Y$ such that $f(x) \in \operatorname{wci}(\varphi(x))$ for each $x \in X$.*

## 2. The role of countable paracompactness for continuous selections avoiding extreme points

The following selection theorem is due to E. Michael [11] and S. Nedev [12].

**Theorem 9** (E. Michael [11, Theorem 3.2′], S. Nedev [12, Theorem 4.2]). *A $T_1$-space $X$ is $\lambda$-collectionwise normal if and only if for every Banach space $Y$ with $w(Y) \leq \lambda$, every l.s.c. mapping $\varphi : X \to \mathcal{C}'_c(Y)$ admits a continuous selection.*

Although the existence itself of a continuous selection is guaranteed by Theorem 9, the assumption in Theorem 4 that $X$ is countably paracompact can not be dropped. Suggested by this fact, we are next concerned with the role of countable paracompactness to obtain a continuous selections avoiding extreme points. Our study has two directions; one is to obtain an l.s.c. set-valued selection avoiding extreme points under a separation axiom of $X$ weaker than $\lambda$-collectionwise normality, another is to drop countable paracompactness instead of imposing a condition to set-valued mappings.

### 2.1. L.s.c. set-valued selections avoiding extreme points.

For a mapping $\varphi : X \to 2^Y$, a mapping $\theta : X \to 2^Y$ is called a *set-valued selection* if $\theta(x) \subset \varphi(x)$ for each $x \in X$. A topological space $X$ is called *countably metacompact* if every countable open cover $\mathcal{U}$ of $X$ is refined by a point-finite open cover of $X$. We have the following characterization of countably metacompact spaces without any separation axiom.

**Theorem 10** ([16]). *A topological space $X$ is countably metacompact if and only if for every normed space $Y$, every l.s.c. mapping $\varphi : X \to \mathcal{C}_c(Y)$ with* $\operatorname{Card}\varphi(x) > 1$ *for each $x \in X$ admits an l.s.c. set-valued selection $\phi : X \to \mathcal{C}_c(Y)$ such that $\phi(x) \subset \operatorname{wci}(\varphi(x))$ for each $x \in X$.*

If the mappings $\varphi, \phi : X \to \mathcal{C}_c(Y)$ can be replaced with mappings $\varphi, \phi : X \to \mathcal{C}'_c(Y)$, then Theorem 4 follows from Theorem 9 and the replaced statement. But the author does not know whether Theorem 10 remains valid even if the mappings $\varphi, \phi : X \to \mathcal{C}_c(Y)$ are replaced with $\varphi, \phi : X \to \mathcal{C}'_c(Y)$.

A topological space $X$ is *almost $\lambda$-expandable* ([9], [14]) if for every locally finite collection $\{F_\alpha \mid \alpha \in A\}$ of closed subsets of $X$ with $\operatorname{Card} A \leq \lambda$, there exists a point-finite collection $\{U_\alpha \mid \alpha \in A\}$ of open subsets of $X$ such that $F_\alpha \subset U_\alpha$ for each $\alpha \in A$. Note that every countably paracompact $\lambda$-collectionwise normal space is almost $\lambda$-expandable ([8]), and every almost $\lambda$-expandable space is countably metacompact ([9, Theorem 2.6]). For compact-valued l.s.c. set-valued selections of mappings $\varphi : X \to \mathcal{C}'_c(Y)$, we have the following.

**Theorem 11** ([16]). *A normal space $X$ is almost $\lambda$-expandable if and only if for every Banach space $Y$ with $w(Y) \leq \lambda$, every l.s.c. mapping $\varphi : X \to \mathcal{C}'_c(Y)$ with* $\operatorname{Card}\varphi(x) > 1$ *for each $x \in X$ admits an l.s.c. set-valued selection $\phi : X \to \mathcal{C}_c(Y)$ such that $\phi(x) \subset \operatorname{wci}(\varphi(x))$ for each $x \in X$.*

A $T_1$-space $X$ is *$\lambda$-PF-normal* if every point-finite open cover is normal. A $T_1$-space $X$ is *PF-normal* if $X$ is $\lambda$-PF-normal for each infinite cardinal $\lambda$. PF-normal spaces are first investigated by E. Michael [10], and the name "PF-normal" is due to J. C. Smith [13]. Note that every $\lambda$-collectionwise normal space is $\lambda$-PF-normal and $\omega$-PF-normality coincides with the normality, where $\omega$ is the first infinite cardinal number. T. Kandô [6] and S. Nedev [12] proved the following selection theorem for $\lambda$-PF-normal spaces (PF-normal spaces are called pointwise-paracompact and normal in [6], while $\lambda$-PF-normal spaces are called $\lambda$-pointwise-$\aleph_0$-paracompact and normal in [12]).

**Theorem 12** (T. Kandô [6, Theorem IV], S. Nedev [12, Theorem 4.1]). *A $T_1$-space $X$ is $\lambda$-PF-normal if and only if for every Banach space $Y$ with $w(Y) \leq \lambda$, every l.s.c. mapping $\varphi : X \to \mathcal{C}_c(Y)$ admits a continuous selection.*

A space is countably paracompact and $\lambda$-collectionwise normal if and only if it is almost $\lambda$-expandable and $\lambda$-PF-normal. Thus Theorem 4 follows from Theorems 11 and Theorem 12. Also, by Theorems 10 and 12, we have the following.

**Theorem 13** ([16]). *A $T_1$-space $X$ is countably paracompact and $\lambda$-PF-normal if and only if for every Banach space $Y$ with $w(Y) \leq \lambda$, every l.s.c. mapping $\varphi : X \to \mathcal{C}_c(Y)$ with* $\operatorname{Card}\varphi(x) > 1$ *for each $x \in X$ admits a continuous selection $f : X \to Y$ such that $f(x) \in \operatorname{wci}(\varphi(x))$.*

A topological space $X$ is called *$\lambda$-metacompact* if every open cover $\mathcal{U}$ of $X$ with $\operatorname{Card}\mathcal{U} \leq \lambda$ is refined by a point-finite open cover of $X$. M. M. Čoban [1, Theorem 6.1] characterized $\lambda$-metacompactness in terms of l.s.c. set-valued selections. For $\lambda$-metacompact analogue of Theorem 11, we have the following.

**Theorem 14** ([16]). *A regular space $X$ is $\lambda$-metacompact if and only if for every Banach space $Y$ with $w(Y) \leq \lambda$, every l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$ with*

$\operatorname{Card} \varphi(x) > 1$ *for each* $x \in X$ *admits an l.s.c. set-valued selection* $\phi : X \to \mathcal{C}_c(Y)$ *such that* $\phi(x) \subset \operatorname{wci}(\varphi(x))$ *for each* $x \in X$.

2.2. **Dropping countable paracompactness.** Next, we drop countable paracompactness of Theorem 4 instead of imposing a condition to set-valued mappings. In fact, the additional condition for set-valued mappings is that the values of them has uniformly large diameters. For a subset $A$ of a metric space $(Y, d)$, let $\operatorname{diam} A = \sup\{d(y_1, y_2) \mid y_1, y_2 \in A\}$.

**Theorem 15** ([16]). *A $T_1$-space $X$ is $\lambda$-collectionwise normal if and only if for every Banach space $Y$ with $w(Y) \leq \lambda$, every l.s.c. mapping $\varphi : X \to \mathcal{C}'_c(Y)$ with* $\inf\{\operatorname{diam} \varphi(x) \mid x \in X\} > 0$ *admits a continuous selection $f : X \to Y$ such that* $f(x) \in \operatorname{wci}(\varphi(x))$ *for each* $x \in X$.

We also have the following characterization of $\lambda$-PF-normal spaces.

**Theorem 16** ([16]). *A $T_1$-space $X$ is $\lambda$-PF-normal if and only if for every Banach space $Y$ with $w(Y) \leq \lambda$, every l.s.c. mapping $\varphi : X \to \mathcal{C}_c(Y)$ with* $\inf\{\operatorname{diam} \varphi(x) \mid x \in X\} > 0$ *admits a continuous selection $f : X \to Y$ such that* $f(x) \in \operatorname{wci}(\varphi(x))$ *for each* $x \in X$.

Let $X$ be a topological space and $(Y, d)$ a metric space. A mapping $\varphi : X \to 2^Y$ is said to be *$d$-upper semicontinuous* (*$d$-u.s.c.* for short) if for each $x \in X$ and $\varepsilon > 0$, there exists a neighborhood $U$ of $x$ such that $\varphi(x') \subset B(\varphi(x), \varepsilon)$ for each $x' \in U$. A mapping $\varphi : X \to 2^Y$ is called *$d$-proximal continuous* if $\varphi$ is l.s.c. and $d$-u.s.c. If $\varphi : X \to 2^Y$ is $d$-proximal continuous for some metric $d$ compatible with the topology of $Y$, then $\varphi$ is called *proximal continuous*. Note that all continuous mappings $f : X \to (\mathcal{F}(Y), \tau_V)$ and $f : X \to (\mathcal{F}(Y), \tau_{H(d)})$ are proximal continuous, where $\tau_V$ is the Vietoris topology on $\mathcal{F}(Y)$ and $\tau_{H(d)}$ is the topology on $\mathcal{F}(Y)$ induced by the Hausdorff distance with respect to some compatible metric $d$ of $Y$ (see [4, Section 2]). V. Gutev [4, Theorem 6.1] proved that for every topological space $X$ and for every Banach space $Y$, every proximal continuous mapping $\varphi : X \to \mathcal{F}_c(Y)$ admits a continuous selection. For continuous selections avoiding extreme points, we have the following.

**Theorem 17** ([16]). *Let $X$ be a topological space, $Y$ a Banach space and $\varphi : X \to \mathcal{F}_c(Y)$ a proximal continuous mapping. Then there exists a continuous selection $f : X \to Y$ of $\varphi$ such that* $f(x) \in \operatorname{wci}(\varphi(x))$ *whenever* $\operatorname{Card} \varphi(x) > 1$.

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